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APPROXIMATION OF ANALYTIC BY BOREL SETS AND DEFINABLE COUNTABLE CHAIN CONDITIONS

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ABSTRACT

Let I be a σ-ideal on a Polish space such that each set from I is contained in a Borel set from I. We say that I fails to fulfil the $\mathbf{\Sigma}^1_1$ countable chain condition if there is a $\mathbf{\Sigma}^1_1$ equivalence relation with uncountably many equivalence classes none of which is in I. Assuming definable determinacy, we show that if the family of Borel sets from I is definable in the codes of Borel sets, then each $\mathbf{\Sigma}^1_1$ set is equal to a Borel set modulo a set from I iff I fulfils the $\mathbf{\Sigma}^1_1$ countable chain condition. Further we characterize the σ-ideals I generated by closed sets that satisfy the countable chain condition or, equivalently in this case, the approximation property for $\mathbf{\Sigma}^1_1$ sets mentioned above. It turns out that they are exactly of the form $MGR(\mathcal{F}) = \{A : \forall F \in \mathcal{F} A \cap F \text{ is meager in } F\}$ for a countable family $\mathcal{F}$ of closed sets. In particular, we verify partially a conjecture of Kunen by showing that the σ-ideal of meager sets is the unique σ-ideal on $\mathbf{R}$, or any Polish group, generated by closed sets which is invariant under translations and satisfies the countable chain condition.

1. Introduction

The main objects of our study will be σ-ideals of subsets of Polish spaces. By a **σ-ideal** on X we mean a family of subsets of X which is closed under taking subsets and countable unions. All σ-ideals considered in this paper are assumed to be **proper**, i.e., they do not contain X, and **uniform**, i.e., they contain all singletons $\{x\}, x \in X$. Here are some other relevant definitions. A σ-ideal

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I is said to be **Borel supported** (**$\mathbf{\Sigma}^0_2$ supported**, resp.) if for any $A \in I$ there is $B \in \mathbf{\Delta}^1_1 \cap I$ ($B \in \mathbf{\Sigma}^0_2 \cap I$, resp.) with $A \subset B$. Note that a σ-ideal is $\mathbf{\Sigma}^0_2$ supported iff it is generated by a family of closed sets. A σ-ideal I has the **approximation property** if for any $A \in \mathbf{\Sigma}^1_1$ there is $B \in \mathbf{\Delta}^1_1$ such that $A \Delta B = (A \setminus B) \cup (B \setminus A) \in I$. Note that, in case I is Borel supported, this is equivalent to saying that if $A \in \mathbf{\Sigma}^1_1$, then there are $B_1, B_2 \in \mathbf{\Delta}^1_1$ such that $B_1 \subset A \subset B_2$ and $B_2 \setminus B_1 \in I$. We say that a σ-ideal I fulfils the **countable chain condition** (the **c.c.c.**) if any family $\mathcal{A}$ of disjoint Borel sets such that $\mathcal{A} \cap I = \emptyset$ is countable. It is well-known that if a Borel supported σ-ideal fulfils the c.c.c., then it has the approximation property (see e.g. the proof of Lemma 5 below). In particular cases, like, e.g., $I =$ the family of meager sets or the family of measure zero sets for some σ-finite Borel measure, this says that analytic sets have the Baire property and are measurable. It also follows from the above fact that, in case I is Borel supported, the members of $\mathcal{A}$ in the definition of the c.c.c. can be assumed to be merely $\mathbf{\Sigma}^1_1$ without changing the meaning of this definition.

Let $\mathcal{A}$ be a family of disjoint sets. One can naturally associate with such a family the equivalence relation $E_{\mathcal{A}}$:

$$xE_{\mathcal{A}}y \Leftrightarrow (\forall A \in \mathcal{A}\, x \in A \Leftrightarrow y \in A). \tag{1}$$

Thus a Borel supported σ-ideal I does not fulfil the c.c.c. iff there is an equivalence relation E with $|X/E| > \omega$ whose equivalence classes, except for possibly one, are $\mathbf{\Sigma}^1_1$ and do not belong to I. We propose the following definable version of the c.c.c. We say that a Borel supported σ-ideal I fulfils the **$\mathbf{\Sigma}^1_1$ c.c.c.** if there is no $\mathbf{\Sigma}^1_1$ equivalence relation E with $|x/E| > \omega$ whose all, but possibly countably many, equivalence classes are not in I. (We get an equivalent version of this definition if we assume that none of the equivalence classes of E is in I.) The main result of the first part of the present paper is that the $\mathbf{\Sigma}^1_1$ c.c.c. is equivalent with the approximation property (assuming some determinacy and definability of the σ-ideal). This gives an answer to a question of Mauldin [M1]. We also define the pseudo-Borel c.c.c. and prove a version of the above result (the pseudo-Borel c.c.c. replacing the $\mathbf{\Sigma}^1_1$ c.c.c.) without assuming any determinacy hypotheses. As a lemma we prove (see Lemma 4) the following result which seems interesting in its own right: Assume $\mathbf{\Delta}^1_2$-determinacy. If E is a $\mathbf{\Sigma}^1_1$ equivalence relation, then E has countably many equivalence classes iff every E-invariant $\mathbf{\Sigma}^1_1$ set is Borel. (After this paper was written, G. Hjorth showed that $\mathbf{\Delta}^1_2$-determinacy can be replaced in the above statement by the assumption that $x^{\#}$ exists for all $x \in \omega^\omega$, which is equivalent, by results of Harrington and Martin, to $\mathbf{\Sigma}^1_1$-determinacy.)

In the second part we examine which $\mathbf{\Sigma}^0_2$ supported σ-ideals fulfil the $\mathbf{\Sigma}^1_1$ c.c.c. It turns out that the $\mathbf{\Sigma}^1_1$ c.c.c. is equivalent in this case with the c.c.c. Actually we show that $\mathbf{\Sigma}^0_2$ supported σ-ideals fulfilling the c.c.c. are of the form $I = \{A : \forall F \in \mathcal{F}\ A \cap F \text{ is meager in } F\}$ for some countable well-ordered by reverse inclusion family $\mathcal{F}$ of closed sets. On the other hand, if the c.c.c. is violated by a $\mathbf{\Sigma}^0_2$ supported σ-ideal I, then there exists a homeomorphic embedding ϕ: $2^\omega \times \omega^\omega \to X$ such that $\phi[\{\alpha\} \times \omega^\omega] \notin I$ for any $\alpha \in 2^\omega$. This sharpens and generalizes some earlier results of Mauldin [M] and Balcerzak, Baumgartner and Hejduk [BBH]. We use this fact to show that if I is a $\mathbf{\Sigma}^0_2$ supported σ-ideal of subsets of a Polish group which is translation invariant and fulfils the c.c.c., then it is the σ-ideal of meager sets. This gives a partial answer to a question of Kunen [KU].

2. Approximating $\mathbf{\Sigma}^1_1$ sets and the $\mathbf{\Sigma}^1_1$ c.c.c.

It is a well-known fact that if a Borel supported σ-ideal fulfills the c.c.c., then it has the approximation property (see Lemma 5 below). That the reverse implication also holds in certain particular cases was proved in [KLW]. A combination of Theorem 7(ii), Proposition 6(ii) of Section 3 in [KLW] yields the following result: Let I be a Borel supported σ-ideal such that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Pi}^1_1$ in the codes of Borel sets and such that for any $A \in \mathbf{\Delta}^1_1 \setminus I$ there exists a closed set $C \notin I$ with $C \subset A$. Then I has the approximation property iff I fulfills the c.c.c. Also Mauldin [M1] proved, using results from [M], that the σ-ideal of subsets of [0, 1] which can be covered by a $\mathbf{\Sigma}^0_2$ set of Lebesgue measure zero (the σ-ideal very strongly violates the c.c.c. as was shown in [M]) does not have the approximation property. Here, using quite different methods and assuming an appropriate amount of determinacy, we are able to prove that the approximation property is actually equivalent to the $\mathbf{\Sigma}^1_1$ c.c.c., for all reasonably definable Borel supported σ-ideals regardless of their other structural properties. This gives an answer to a question of Mauldin [M1], who asked what properties of a σ-ideal are responsible for it having the approximation property.

If E is an equivalence relation on X and $A \subset X$ is E-invariant, we write $|A/E|$ for the cardinality of the family of equivalence classes included in A. If $B \subset X$, then $[B]_E$ denotes the saturation of B with respect to E, i.e., $[B]_E =$

$\{x \in X : \exists y \in B\, xEy\}$. We write $[x]_E$ for $[\{x\}]_E$. If there is no possibility of confusion we will drop the subscript E. If σ and τ are two sequences of elements of a set Y then $\sigma * \tau$ denotes their concatenation. If $y \in Y$, then $\sigma * y = \sigma * (0, y)$. For a definition of $\mathbf{\Pi}^1_1$-rank see [K1, 34B]. Now we define the set $WO \subset 2^\omega$. Let $\langle\ ,\ \rangle : \omega^2 \to \omega$ be a bijection. Put $\alpha \in WO$ iff the relation $\{(n,m) \in \omega^2 : \alpha(\langle n,m\rangle) = 1\}$ well orders ω. WO is $\mathbf{\Pi}^1_1$. Define $|\alpha|$ = the order type of $\{(n,m) \in \omega^2 : \alpha(\langle n,m\rangle) = 1\}$ for $\alpha \in WO$. Then $\alpha \to |\alpha|$ is a $\mathbf{\Pi}^1_1$-rank on WO. For a pointclass Γ, $\mathrm{Det}(\Gamma)$ means that all games in Γ are determined. By $\sigma(\mathbf{\Pi}^1_2)$ we denote the σ-algebra generated by the family of all $\mathbf{\Pi}^1_2$ sets.

THEOREM 1: *Assume* $\mathrm{Det}(\mathbf{\Delta}^1_2)$. *Let I be a Borel supported σ-ideal such that the family $I \cap \mathbf{\Delta}^1_1$ is $\sigma(\mathbf{\Pi}^1_2)$ in the codes of Borel sets. Then I has the approximation property iff I fulfils the $\mathbf{\Sigma}^1_1$ c.c.c.*

The proof of the theorem is split up into several lemmas. The implication $\Rightarrow$ follows from Lemmas 3 and 4 and the implication $\Leftarrow$ follows from Lemmas 5 and 6. Note that the assumption that $I \cap \mathbf{\Delta}^1_1$ is $\sigma(\mathbf{\Pi}^1_2)$ in the codes is used only in the proof of $\Leftarrow$.

The following consequence of Theorem 4 from [KW] will be useful.

LEMMA 1: (Kechris–Woodin) $\mathrm{Det}(\mathbf{\Delta}^1_2)$ *implies* $\mathrm{Det}(\sigma(\mathbf{\Pi}^1_2))$.

We will be also using the following particular case of a theorem due to Solovay. For a proof see [K, Theorem 7.1].

LEMMA 2: (Solovay) *Assume* $\mathrm{Det}(\mathbf{\Delta}^1_2)$. *Let A be a $\mathbf{\Pi}^1_1$ set and ρ a $\mathbf{\Pi}^1_1$-rank on A. Let $B \subset A$ be $\sigma(\mathbf{\Pi}^1_2)$ and such that if $\rho(x) = \rho(y)$ and $x \in B, y \in A$ then $y \in B$. Then $B \in \mathbf{\Pi}^1_1$.*

LEMMA 3: *Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation whose all but countably many classes are not in I. Let A be an E-invariant set. If $A \notin \mathbf{\Delta}^1_1$, then there is no $B \in \mathbf{\Delta}^1_1$ such that $A \Delta B \in I$.*

Proof: Assume otherwise. Since I is Borel supported, we can suppose that there are Borel sets C and D such that $C \cap A = \emptyset, D \subset A$ and $X \setminus (C \cup D) \in I$. Now, $[C]$ and $[D]$ are $\mathbf{\Sigma}^1_1$ and also $[C] \cap A = \emptyset$ and $[D] \subset A$, as A is E-invariant. Let $\{O_n\colon\ n \in \omega\}$ be the family of all equivalence classes of E which are in I. Each O_n is $\mathbf{\Sigma}^1_1$. If $[C] \cup [D] \cup \bigcup_{n\in\omega} O_n = X$, then, since A is E-invariant, $A = [D] \cup \bigcup_{O_n \subset A} O_n$ and $X \setminus A = [C] \cup \bigcup_{O_n \cap A = \emptyset} O_n$. Now, the Suslin theorem implies that A is Borel which contradicts the assumptions. Thus there exists

$x \in X \setminus ([C] \cup [D] \cup \bigcup_{n\in\omega} O_n)$. Then $[x] \notin I$ and $[x] \subset X \setminus (C \cup D) \in I$, a contradiction. ∎

LEMMA 4: *Assume* $\mathrm{Det}(\mathbf{\Delta}^1_2)$. *Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation. If E has uncountably many equivalence classes, then there exists an E-invariant set $A \in \mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1$. (Thus E has countably many equivalence classes iff every E-invariant $\mathbf{\Sigma}^1_1$ set is Borel.)*

Proof: Assume that such an A does not exist. Then $[A] \in \mathbf{\Delta}^1_1$ for any $A \in \mathbf{\Sigma}^1_1$. We claim that either there exists a Borel uncountable set $C \subset X$ such that xEy iff $x = y$ for $x, y \in C$, or there exists an E-invariant set $B \in \mathbf{\Delta}^1_1$ such that $|B/E| > \omega$ and if $B' \subset B$ is $\mathbf{\Delta}^1_1$ and E-invariant then $|B'/E| \leq \omega$ or $|(B \setminus B')/E| \leq \omega$. (The proof below is related to arguments of Becker [B], Sami and Stern on minimal counterexamples to the Vaught conjecture.) To prove this assume that for any E-invariant $B \in \mathbf{\Delta}^1_1$ there exist E-invariant $\mathbf{\Delta}^1_1$ sets $B_1, B_2 \subset B$ such that $B_1 \cap B_2 = \emptyset$ and $|B_1/E| > \omega$, $|B_2/E| > \omega$. We construct a countable Boolean algebra $\mathcal{A}$ of Borel sets such that:

(i) $\mathcal{A}$ contains a countable topological basis of X;

(ii) if $B \in \mathcal{A}$ and $|[B]/E| > \omega$ then there exist $B_1, B_2 \in \mathcal{A}$ such that $B_1, B_2 \subset B$, $[B_1] \cap [B_2] = \emptyset$, and $|[B_1]/E| > \omega$, $|[B_2]/E| > \omega$;

(iii) the topology generated by $\mathcal{A}$ is Polish.

$\mathcal{A}$ is built recursively starting from a countable topological basis of X. We easily take care of (ii) using the assumption on E. To get (iii), we apply two well-known facts: a topology on a standard Borel space can be extended by Borel sets to obtain a Polish topology (see [K1, Theorem 13.1]), and an increasing union of Polish topologies is Polish (see [K1, Lemma 13.3]).

Now we fix a complete metric d on X which is compatible with the topology generated by $\mathcal{A}$, and do a Cantor-type construction producing open (in this topology) sets $Q_\sigma, \sigma \in 2^{<\omega}$, so that:

(a) $Q_\emptyset = X$;

(b) $d\text{-diam}(Q_\sigma) \leq 1/(lh\sigma + 1)$;

(c) $|[Q_\sigma]/E| > \omega$;

(d) $d\text{-closure}(Q_{\sigma * i}) \subset Q_\sigma$ for $i \in 2$ and $\sigma \in 2^{<\omega}$;

(e) if $\sigma, \tau \in 2^{<\omega}$ are incompatible, then $[Q_\sigma] \cap [Q_\tau] = \emptyset$.

When Q_σ, for some $\sigma \in 2^{<\omega}$, has been constructed, we find by (ii) open (in the topology generated by $\mathcal{A}$) sets $U_0, U_1 \subset Q_\sigma$ such that $|[U_i]/E| > \omega, i = 1, 2$, and

$[U_0] \cap [U_1] = \emptyset$. Now for $i = 1, 2$ find $V_n^i, n \in \omega$, such that V_n^i is open in the topology generated by $\mathcal{A}$, d-closure$(V_n^i) \subset Q_\sigma$, d-diam$(V_n^i) < 1/(lh\sigma + 2)$ and $\bigcup_{n\in\omega} V_n^i = U_i$. Then $|[V_{n_i}^i]/E| > \omega$ for some $n_i \in \omega$. Put $Q_{\sigma * i} = V_{n_i}^i$ for $i = 1, 2$.

Now $C = \bigcap_{n\in\omega} \bigcup_{lh\sigma = n} Q_\sigma$ is an uncountable Borel (in the original topology) set whose distinct elements lie in distinct equivalence classes of E.

If there exists an uncountable Borel set C as above, we can find a $\mathbf{\Sigma}_1^1$ non-Borel set $A \subset C$. Then $[A] \cap C = A$, whence $[A] \notin \mathbf{\Delta}_1^1$, a contradiction.

Thus we can assume, by passing to a Borel invariant subset of X, that $|X/E| > \omega$ and for each $\mathbf{\Sigma}_1^1$ set $A \subset X, |[A]/E| \leq \omega$ or $|(X \setminus [A])/E| \leq \omega$. Using Det$(\mathbf{\Pi}_1^1)$, by Burgess' theorem [Bu], there exists a $\mathbf{\Delta}_2^1$ function $f : X \to WO$ such that $xEy \Leftrightarrow |f(x)| = |f(y)|$. Put $B = \{x \in WO : \exists y \in X \, |f(y)| = |x|\}$. Then $B \in \mathbf{\Sigma}_2^1$ and fulfils the assumptions of Lemma 2 (with $A = WO$ and $\rho(x) = |x|$). Thus $B \in \mathbf{\Pi}_1^1$. Now define

$$B' = \{x \in B : \exists z \in B \, (|z| < |x| \wedge \forall y \, (y \in B \wedge |y| < |x| \Rightarrow |y| \leq |z|))\}.$$

It follows that $B' \in \mathbf{\Sigma}_2^1$. Put $A = f^{-1}(B')$. Then $A \in \mathbf{\Sigma}_2^1$ and is E-invariant. Also A as well as its complement contain uncountably many equivalence classes of E. Thus $A \in \mathbf{\Sigma}_2^1 \setminus \mathbf{\Pi}_1^1$. By Det$(\mathbf{\Delta}_2^1)$ and Lemma 1, each $\mathbf{\Sigma}_1^1$ set is Borel reducible to A. Pick $D \subset 2^\omega$ with $D \in \mathbf{\Sigma}_1^1 \setminus \mathbf{\Delta}_1^1$. Let $\phi : 2^\omega \to X$ be Borel and such that $x \in D \Leftrightarrow \phi(x) \in A$. Since A is E-invariant, $x \in D \Leftrightarrow \phi(x) \in [\phi[D]] \in \mathbf{\Delta}_1^1$. Thus D is $\mathbf{\Delta}_1^1$, a contradiction. ∎

LEMMA 5: *If I does not have the approximation property, then there exists a $\mathbf{\Pi}_1^1$ set A with a $\mathbf{\Pi}_1^1$-rank ρ such that the set $T \subset \omega_1$ defined by $\alpha \in T$ iff $\{x : \rho(x) = \alpha\} \notin I$ is uncountable.*

Proof: Let P be a $\mathbf{\Sigma}_1^1$ set such that there is no $B \in \mathbf{\Delta}_1^1$ with $P \Delta B \in I$. Then the same is true about the $\mathbf{\Pi}_1^1$ set $Q = X \setminus P$. Let ϕ be a Borel mapping from X to the space of all trees on ω such that $\phi(x)$ is well founded iff $x \in Q$. For a tree T on ω and $u \in \omega^{<\omega}$, put $T_u = \{v \in \omega^{<\omega} : u * v \in T\}$. If T is well founded, let $|T|$ denote the rank of T. Suppose $\forall u \in \omega^{<\omega} \exists \xi < \omega_1 \forall \zeta > \xi \{x : \phi(x)_u$ is well founded and $|\phi(x)_u| = \zeta\} \in I$. Then for each $u \in \omega^{<\omega}$ there exists a smallest $\xi = \xi_u < \omega_1$ as above. Put $\bar{\xi} = \sup\{\xi_u : u \in \omega^{<\omega}\} + 1$. Now define $B = \{x \in X : \phi(x)$ is well founded and $|\phi(x)| \leq \bar{\xi}\}$ and $B' = \{x \in X : \exists u \in \omega^{<\omega} \phi(x)_u$ is well founded and $|\phi(x)_u| = \bar{\xi}\}$. Then it is easy to check that $B \subset Q \subset B \cup B'$, $B, B' \in \mathbf{\Delta}_1^1$ and $B' \in I$ which contradicts our assumption on Q. Thus there exists $\bar{u} \in \omega^{<\omega}$ such that $\forall\xi < \omega_1 \exists\zeta > \xi\{x \in X : \phi(x)_{\bar{u}}$ is well founded and $|\phi(x)_{\bar{u}}| = \zeta\} \notin I$. Put $A = \{x \in X : \phi(x)_{\bar{u}}$ is well founded$\}$ and $\rho(x) = |\phi(x)_{\bar{u}}|$. It is easy to verify that these A and ρ work. ∎

LEMMA 6: *Assume* $\mathrm{Det}(\boldsymbol{\Delta}^1_2)$. *Let* $I \cap \boldsymbol{\Delta}^1_1$ *be* $\sigma(\boldsymbol{\Pi}^1_2)$ *in the codes. If I does not have the approximation property, then there is a* $\boldsymbol{\Sigma}^1_1$ *equivalence relation E such that* $|X/E| = \omega_1$ *and all equivalence classes of E, except for perhaps one, are not in I.*

Proof: Take A and ρ as in Lemma 5. Define $A' = \{x \in A : \{y \in A : \rho(y) = \rho(x)\} \notin I\}$. Since $I \cap \boldsymbol{\Delta}^1_1$ is $\sigma(\boldsymbol{\Pi}^1_2)$ in the codes, A' is $\sigma(\boldsymbol{\Pi}^1_2)$. Clearly A' fulfils the assumption of Lemma 2 whence $A' \in \boldsymbol{\Pi}^1_1$. Then the following equivalence relation is $\boldsymbol{\Sigma}^1_1$:

$$xEy \Leftrightarrow ((x \in A' \vee y \in A') \Rightarrow (x \in A' \wedge y \in A' \wedge \rho(x) = \rho(y))).$$

Also E has ω_1 equivalence classes and all of them except for perhaps $X \setminus A'$ are not in I. ∎

Assuming more determinacy and using the full strength of Solovay's lemma (see [K, Theorem 7.1]) we obtain the same conclusion (with the same proof) as in Theorem 1 for wider classes of Borel supported σ-ideals or even for all of them if we assume AD. (Note however that, as follows from Lemmas 2 and 3, it is enough to have only $\mathrm{Det}(\boldsymbol{\Delta}^1_2)$ to prove that the approximation property implies the $\boldsymbol{\Sigma}^1_1$ c.c.c. for all Borel supported σ-ideals.) For example we have the following result.

THEOREM 1': *Assume PD (AD, resp.). Let I be a Borel supported σ-ideal such that* $I \cap \boldsymbol{\Delta}^1_1$ *is projective in the codes* ($I \cap \boldsymbol{\Delta}^1_1$ *is arbitrary, resp.). Then the* $\boldsymbol{\Sigma}^1_1$ *c.c.c. and the approximation property are equivalent.*

We want to make here a few comments on what can be proved without any determinacy hypotheses. We will summarize them in Theroem 1". A family $\mathcal{A}$ of disjoint sets is called **pseudo-Borel** if the relation $E_{\mathcal{A}}$ associated with $\mathcal{A}$ as in (1) in the Introduction is $\boldsymbol{\Sigma}^1_1$ and there is a $\boldsymbol{\Pi}^1_1$ equivalence relation F such that

$$x \in \bigcup \mathcal{A} \Rightarrow (\forall y\, xFy \Leftrightarrow xE_{\mathcal{A}}y). \tag{2}$$

Note that if $E_{\mathcal{A}}$ is Borel we can take $F = E_{\mathcal{A}}$. A Borel supported σ-ideal I fulfils the **pseudo-Borel c.c.c.** if every pseudo-Borel family $\mathcal{A}$ of disjoint sets such that $\mathcal{A} \cap I = \emptyset$ is countable. Clearly the c.c.c. implies the $\mathbf{\Sigma}^1_1$ c.c.c., which in turn implies the pseudo-Borel c.c.c.

LEMMA 7: *Assume a Borel supported σ-ideal has the approximation property. Then I fulfils the pseudo-Borel c.c.c.*

Proof: Suppose I does not fulfil the pseudo-Borel c.c.c. Let $\mathcal{A}$ be a pseudo-Borel family of sets witnessing it and let F be a $\mathbf{\Pi}^1_1$ equivalence relation from the definition of pseudo-Borelness. By Lemma 3 applied to $E_{\mathcal{A}}$ it is enough to find an $E_{\mathcal{A}}$-invariant set A such that $A \in \mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1$. Since $E_{\mathcal{A}} \in \mathbf{\Sigma}^1_1, X \setminus \bigcup \mathcal{A} \in \mathbf{\Sigma}^1_1$. If $X \setminus \bigcup \mathcal{A} \notin \mathbf{\Delta}^1_1$ we are done. Thus we can assume that $\bigcup \mathcal{A} \in \mathbf{\Delta}^1_1$. But by (2) $\bigcup \mathcal{A}$ is F-invariant and $F | \bigcup \mathcal{A} = E_{\mathcal{A}} | \bigcup \mathcal{A}$. Thus since $F \in \mathbf{\Pi}^1_1$ and $|\bigcup \mathcal{A}/F| = |\mathcal{A}| > \omega$, by Silver's theorem [S], there is a perfect compact set $C \subset \bigcup \mathcal{A}$ such that different elements of C belong to different equivalence classes of $E_{\mathcal{A}}$. Pick $A \subset C$ in $\mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1$. Then $[A]_{E_{\mathcal{A}}}$ is $E_{\mathcal{A}}$-invariant and $\mathbf{\Sigma}^1_1$ and, as $[A]_{E_{\mathcal{A}}} \cap C = A, [A]_{E_{\mathcal{A}}} \notin \mathbf{\Delta}^1_1$. ∎

LEMMA 8: *Assume I is a Borel supported σ-ideal such that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes of Borel sets. If I fulfils the pseudo-Borel c.c.c., then I has the approximation property.*

Proof: It is enough to prove an analogue of Lemma 6 without the determinacy hypothesis. But since we assume that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes, the set A' defined in the proof of Lemma 6 is $\mathbf{\Pi}^1_1$. Put $\mathcal{A} = \{\{x \in A' : \rho(x) = \alpha\} : \alpha < \omega_1\}$. Then $E_{\mathcal{A}}$ is equal to the relation E defined in the proof of Lemma 6 and thus $|X/E_{\mathcal{A}}| > \omega$ and $E_{\mathcal{A}} \in \mathbf{\Sigma}^1_1$. For the $\mathbf{\Pi}^1_1$ equivalence relation F we take

$$xFy \Leftrightarrow (x = y \vee (x \in A' \wedge y \in A' \wedge \rho(x) = \rho(y)))\,.$$

∎

Combining Lemmas 7 and 8 we obtain the following theorem.

THEOREM1'': *Let I be a Borel supported σ-ideal such that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes of Borel sets. Then I has the approximation property iff I fulfils the pseudo-Borel c.c.c.*

3. $\mathbf{\Sigma_2^0}$ supported σ-ideals

The $\mathbf{\Sigma_2^0}$ supported σ-ideals occur frequently in harmonic analysis and descriptive set theory as σ-ideals generated by families of closed sets. In this section we characterize those $\mathbf{\Sigma_2^0}$ supported σ-ideals which have the approximation property and also give an abstract characterization of the σ-ideal of meager sets. No determinacy assumptions will be used in the sequel.

Let $\mathcal{F}$ be a family of subsets of a Polish space X. Put

$$MGR(\mathcal{F}) = \{B \subset X : \forall A \in \mathcal{F}\; B \cap A \text{ is meager in } A\}.$$

If $A \subset X$, we will write $MGR(A)$ for $MGR(\{A\})$. If I is a σ-ideal and $A \subset X$, we write $I|A = \{B \subset A : B \in I\}$. A family $\mathcal{F}$ of subsets of X is said to be **well-ordered by reverse inclusion** if there is an ordinal α such that $\mathcal{F} = \{A_\xi : \xi < \alpha\}$ and $\xi \le \zeta < \alpha \Leftrightarrow A_\xi \supset A_\zeta$. By π_X and π_Y we denote the projections from $X \times Y$ onto X and Y, respectively. Also for $A \subset X \times Y$ we write $A_x = \{y \in Y : (x, y) \in A\}$.

Lemma 9: *Let Y be Polish and let J be a $\mathbf{\Sigma_2^0}$ supported σ-ideal. Assume that for any open set $U \neq \emptyset$ there exists a nowhere dense set $F \subset U$ such that $F \notin J$. Then there is a homeomorphic embedding $\phi : 2^\omega \times \omega^\omega \to Y$ such that $\phi[\{\alpha\} \times \omega^\omega] \notin J$ for any $\alpha \in 2^\omega$.*

Proof: For any family $\mathcal{A}$ of subsets of Y define $\mathcal{A}^d$ to be the set of all points $x \in Y$ such that for any open U with $x \in U$ the set $\{A \in \mathcal{A} : A \cap U \neq \emptyset\}$ is infinite. In the natural way we identify a sequence $\sigma \in (2 \times \omega)^n$ with the sequence $((\sigma)_0, (\sigma)_1) \in 2^n \times \omega^n$. For $\alpha \in \omega^\omega$ by $\alpha|n$ we denote the restriction of α to $n = \{0, \ldots, n-1\}$. We also write $N_\sigma = \{\gamma \in 2^\omega \times \omega^\omega : \pi_{2^\omega}(\gamma)|n = (\sigma)_0, \pi_{\omega^\omega}(\gamma)|n = (\sigma)_1\}$ for $\sigma \in (2 \times \omega)^n, n \in \omega$.

Now we construct recursively open sets $U_\sigma, \sigma \in (2 \times \omega)^{<\omega}$, so that:

(i) $\sigma \subset \tau, \sigma \neq \tau$ implies closure$(U_\tau) \subset U_\sigma$;

(ii) if neither $\sigma \subset \tau$ nor $\tau \subset \sigma$ then $U_\sigma \cap U_\tau = \emptyset$;

(iii) $\operatorname{diam}(U_\sigma) \le 1/2^{n+(\sigma)_1(n-1)}$, where $n = lh\sigma$;

(iv) $\{U_{\sigma*(i,n)} : n \in \omega\}^d \notin J$ for $i \in 2$;

(v) $U_\sigma \neq \emptyset$.

If U_σ has been defined, find a nowhere dense closed set $F \subset U_\sigma$ with $F \notin J$. Then find two closed sets $F_0, F_1 \subset F$, $F_0, F_1 \notin J$ such that there exist two open sets $V_0, V_1 \subset U_\sigma$ containing F_0 and F_1, respectively, and having disjoint closures.

Since F_i is nowhere dense in $V_i, i = 0, 1$, we can find nonempty pairwise disjoint open sets $W_n^i, n \in \omega$, so that $F_i = \{W_n^i : n \in \omega\}^d, W_n^i \subset V_i$ and $\mathrm{diam}(W_n^i) \leq 1/2^{k+1+n}$, where $k = lh\sigma$. To define W_n^i, first choose $D^i = \{d_n^i : n \in \omega\}$ to be discrete subsets of V_i such that $F_i = \mathrm{closure}(D^i) \setminus D^i$. Then let W_n^i be an appropriately small ball around d_n^i. Put $U_{\sigma*(i,n)} = W_n^i$.

Now define $\phi : 2^\omega \times \omega^\omega \to Y$ by $\phi(\alpha, \beta) =$ the only point in $\bigcap_{n\in\omega} U_{(\alpha|n,\beta|n)}$. It is clear from (i)–(iii) and (v) that ϕ is a homeomorphic embedding. Note also that, by (iii) and (iv), $\{\phi[N_{\sigma*(i,n)}] : n \in \omega\}^d = \{U_{\sigma*(i,n)} : n \in \omega\}^d \notin J$ for any $\sigma \in (2 \times \omega)^{<\omega}$ and $i \in 2$.

Suppose that there is $\alpha \in 2^\omega$ such that $\phi[\{\alpha\} \times \omega^\omega] \in J$. Then there exist $F_n \in J \cap \mathbf{\Pi}_1^0, n \in \omega$, such that $\phi[\{\alpha\} \times \omega^\omega] \subset \bigcup_{n\in\omega} F_n$. By the Baire Category Theorem there is $\tau \in \omega^k$, for some $k \in \omega$, and $n_0 \in \omega$ such that $\phi[N_{(\alpha|k,\tau)}] \subset F_{n_0}$. But then $\{\phi[N_{\alpha|(k+1),\tau*(n)}] : n \in \omega\}^d \subset F_{n_0} \in J$, a contradiction. ∎

The following theorem generalizes and strengthens some results proved in [M] and [BBH]. It was shown in [BBH, Theorem 2.3] that (ii) holds for the σ-ideal of all subsets of 2^ω which can be covered by $\mathbf{\Sigma}_2^0$ sets of Lebesgue measure zero. A bit weaker result for the same σ-ideal was proved earlier in [M, Theorem 1] and this weaker result was generalized in [BBH, Theorem 1.5] to a slightly wider class of $\mathbf{\Sigma}_2^0$ supported σ-ideals.

THEOREM 2: *Let I be a $\mathbf{\Sigma}_2^0$ supported σ-ideal. Then precisely one of the following possibilities holds:*
(i) *$I = MGR(\mathcal{F})$ for a countable family $\mathcal{F}$ of closed subsets of X, which can be assumed to be well-ordered by reverse inclusion;*
(ii) *there is a homeomorphic embedding $\phi : 2^\omega \times \omega^\omega \to X$ such that $\phi[\{\alpha\} \times \omega^\omega] \notin I$ for any $\alpha \in 2^\omega$.*

Proof: For $F \subset X$ closed put $F' = F \setminus \bigcup\{U : U$ is open, $U \cap F \neq \emptyset$ and $I|(U \cap F) = MGR(U \cap F)\}$ and $F^* = F \setminus \bigcup\{U : U$ is open and $U \cap F \in I\}$. Now define by transfinite recursion:

$F_0 = X^*$;
$F_\lambda = (\bigcap_{\gamma<\lambda} F_\gamma)^*$ if λ is limit;
$F_{\gamma+1} = F'_\gamma$.

Claim: Let $U \subset X$ be open. Assume $F_{\gamma+1} \cap U = F_\gamma \cap U$. Then $F_\xi \cap U = F_\gamma \cap U$ for any $\xi > \gamma$.

Proof of the Claim: First we prove that if $W \cap F_\gamma \in I$ for an open set W, then $W \cap F_\gamma = \emptyset$. This is clear if γ is limit or 0. Assume γ is a successor. Let λ be the biggest limit ordinal $\leq \gamma$ or $\lambda = 0$. Then $W \cap F_\gamma$ must be meager in F_λ. So there exists a biggest $\theta < \gamma$ with $W \cap F_\gamma$ meager in F_θ. It follows that there exists an open set V such that $\emptyset \neq V \cap F_{\theta+1} \subset W \cap F_\gamma$. We thus have $V \cap F_{\theta+1} \in MGR(F_\theta)$ and $V \cap F_{\theta+1} \in I$, whence $V \cap F_{\theta+1} = \emptyset$, a contradiction.

Now, if $U \cap F_\gamma \subset F_{\gamma+1}$, we show by induction on $\xi > \gamma$ that $U \cap F_\gamma \subset F_\xi$. For ξ limit it is a consequence of the observation from the previous paragraph. For successors it follows directly from the inductive hypothesis and the inclusion $U \cap F_\gamma \subset F_{\gamma+1}$. This finishes the proof of the Claim.

There exists a smallest $\alpha < \omega_1$ such that $F_\alpha = F_{\alpha+1}$.

CASE 1: $F_\alpha = \emptyset$.

Put $\mathcal{F} = \{F_\gamma : \gamma < \alpha\}$. First notice that $F_{\gamma+1}$ is nowhere dense in F_γ for $\gamma < \alpha$. Otherwise there is an open set U such that $F_{\gamma+1} \supset F_\gamma \cap U \neq \emptyset$. Then by the Claim $F_\xi \supset F_\gamma \cap U$ for all $\xi > \gamma$. In particular, $F_\alpha \supset F_\gamma \cap U \neq \emptyset$ which contradicts our assumption on F_α.

Now we show that $I = MGR(\mathcal{F})$. Let $A \in I$. Then $A \cap (F_\gamma \setminus F_{\gamma+1}) \in MGR(F_\gamma \setminus F_{\gamma+1})$ for $\gamma < \alpha$. But since $F_{\gamma+1} \in MGR(F_\gamma)$, we have $A \in MGR(F_\gamma)$. For the opposite direction assume that $A \cap F_\gamma \in MGR(F_\gamma)$. Since $F_{\gamma+1}$ is closed, $A \cap (F_\gamma \setminus F_{\gamma+1}) \in MGR(F_\gamma \setminus F_{\gamma+1})$. Thus $A \cap (F_\gamma \setminus F_{\gamma+1}) \in I$ for $\gamma < \alpha$. Also clearly $X \setminus F_0 \in I$ and $\bigcap_{\gamma<\lambda} F_\gamma \setminus F_\lambda \in I$ for λ limit. Since I is a σ-ideal,

$$A = A \cap (X \setminus F_0) \cup \bigcup_{\lambda<\alpha, \lambda\ limit} A \cap (\bigcap_{\gamma<\lambda} F_\gamma \setminus F_\lambda) \cup \bigcup_{\gamma<\alpha} A \cap (F_\gamma \setminus F_{\gamma+1}) \in I.$$

CASE 2: $F_\alpha \neq \emptyset$.

By the Claim $F_\alpha = F_\xi$ for all $\xi > \alpha$. Thus $F'_\alpha = F_\alpha$ and $F^*_\alpha = F_\alpha$. This easily implies that the assumptions of Lemma 9 are fulfilled for $Y = F_\alpha$ and $J = I|F_\alpha$. Thus we obtain (ii). ∎

Note that (i) implies that I fulfils the c.c.c. Thus it follows from Theorem 2 that if a $\mathbf{\Sigma}^0_2$ supported σ-ideal does not fulfil the c.c.c., then there exists a "perfect" family of G_δ's outside of I, i.e., (ii) holds. A similar fact was proved for a different class of σ-ideals in [KLW]. Namely by Theorem 2 of Section 3 in

[KLW], if I is a Borel supported σ-ideal such that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Pi}^1_1$ in the codes and for any $A \in \mathbf{\Delta}^1_1 \setminus I$ there is a closed set $C \notin I$ with $C \subset A$, then if I does not fulfil the c.c.c., then there is a "perfect" family of closed sets not in I. In particular, in this case, as well as in the case of $\mathbf{\Sigma}^0_2$ supported σ-ideals, the c.c.c., the $\mathbf{\Sigma}^1_1$ c.c.c., and the pseudo-Borel c.c.c. are equivalent.

The next theorem lists a few characterizations of the σ-ideals of the form $MGR(\mathcal{F})$ for a countable, well-ordered by reverse inclusion family $\mathcal{F}$ of closed sets.

THEOREM 3: *Let I be a $\mathbf{\Sigma}^0_2$ supported σ-ideal. Then the following are equivalent.*
(i) *I is of the form $MGR(\mathcal{F})$ for a countable family $\mathcal{F}$ of closed subsets of X well-ordered by reverse inclusion;*
(ii) *I fulfils the c.c.c.;*
(iii) *I fulfils the pseudo-Borel c.c.c.;*
(iv) *$I \cap \mathbf{\Delta}^1_1$ is Δ^1_1 in the codes of Borel sets;*
(v) *$I \cap \mathbf{\Delta}^1_1$ is Σ^1_1 in the codes of Borel sets;*
(vi) *I has the approximation property.*

Proof: (i) $\Rightarrow$ (ii). Let $\mathcal{A}$ be an uncountable family of disjoint $\mathbf{\Sigma}^1_1$ sets with $\mathcal{A} \cap I = \emptyset$. Then, since $\mathcal{F}$ is countable, there is $F \in \mathcal{F}$ and an uncountable family $\mathcal{A}' \subset \mathcal{A}$ such that $A \cap F$ is not meager in F for any $A \in \mathcal{A}'$. This yields a contradiction, since $MGR(F)$ fulfils the c.c.c.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Suppose (i) does not hold. Let ϕ be as in Theorem 2(ii). Put $\mathcal{A} = \{\phi[\{\alpha\} \times \omega^\omega] : \alpha \in 2^\omega\}$. Then $E_\mathcal{A}$ is Borel. Indeed, notice that since ϕ is a homeomorphic embedding $\phi[2^\omega \times \omega^\omega]$ is $\mathbf{\Pi}^0_2$. Put $B = \phi[2^\omega \times \omega^\omega]$. Then

$$\begin{aligned} xE_\mathcal{A}y &\Leftrightarrow ((x \notin B \wedge y \notin B) \vee (\exists \alpha \in 2^\omega x, y \in \phi[\{\alpha\} \times \omega^\omega])) \\ &\Leftrightarrow ((x \notin B \wedge y \notin B) \vee (\exists! \alpha \in 2^\omega x, y \in \phi[\{\alpha\} \times \omega^\omega])). \end{aligned}$$

Since $E_\mathcal{A}$ is Borel, $\mathcal{A}$ is a pseudo-Borel family.

(i) $\Rightarrow$ (iv). By a standard calculation, see e.g. [K, 16.1].

(iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (i). Suppose that I is not of the required form. Let ϕ be as in Theorem 2(ii). Let $B \subset \omega^\omega \times 2^\omega$ be such that $B \in \Delta^1_1$ and $\pi_{\omega^\omega}[B] \notin \Pi^1_1$. Define $B' \subset \omega^\omega \times X$ by $(\alpha, x) \in B' \Leftrightarrow x \in \phi[2^\omega \times \omega^\omega] \wedge \big(\alpha, \pi_{2^\omega}(\phi^{-1}(x))\big) \in B$. Clearly $B' \in \Delta^1_1$. It is easy to check that $B'_\alpha \notin I$ or $B'_\alpha = \emptyset$ for any $\alpha \in \omega^\omega$ and

$\{\alpha \in \omega^\omega : B'_\alpha \notin I\} = \{\alpha \in \omega^\omega : B_\alpha \neq \emptyset\} = \pi_{\omega^\omega}[B] \notin \mathbf{\Pi}^1_1$. Thus $\{\alpha \in 2^\omega : B'_\alpha \in I\} \notin \mathbf{\Sigma}^1_1$ which gives a contradiction since if $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes, then $\{\alpha \in 2^\omega : A_\alpha \in I\}$ is $\mathbf{\Sigma}^1_1$ for any Borel set $A \subset 2^\omega \times X$.

(vi) $\Rightarrow$ (iii) is simply Lemma 7.

((iii) $\wedge$ (v)) $\Rightarrow$ (vi) is Lemma 8. ∎

Consider now 2^ω as a group with the coordinatewise addition modulo 2. Kunen [Ku, 1.27] asked if all Borel supported σ-ideals on 2^ω which are translation invariant and fulfil the c.c.c. are: the family of meager sets, the family of Lebesgue measure zero sets or the intersection of the two. The following corollary provides a partial answer to this question.

COROLLARY: *Let X be a Polish space and let H be a group of homeomorphisms of X such that $\bigcup_{h\in H} h[U] = X$ for any open nonempty set $U \subset X$. Let I be a $\mathbf{\Sigma}^0_2$ supported σ-ideal on X. If I fulfils the c.c.c. and is such that $h[A] \in I$ if $A \in I$, then I is the σ-ideal of meager sets. In particular, if G is a Polish group and I is a $\mathbf{\Sigma}^0_2$ supported translation invariant σ-ideal on G which fulfils the c.c.c., then I is the σ-ideal of meager sets.*

Proof: First notice that, by invariance under homeomorphisms from H, I cannot contain a nonempty open set. By Theorem 3 there is a well-ordered by reverse inclusion countable family $\mathcal{F}$ of closed subsets of X such that $I = MGR(\mathcal{F})$. Let $F_0 \in \mathcal{F}$ be such that $F' \subset F_0$ for any $F' \in \mathcal{F}$. Then $X \setminus F_0$ is open and $X \setminus F_0 \in I$. Thus $X \setminus F_0 = \emptyset$, i.e., $F_0 = X$. If $\mathcal{F} \neq \{F_0\}$, let $F_1 \in \mathcal{F}$ be such that $F' \subset F_1$ for any $F' \in \mathcal{F} \setminus \{F_0\}$. If $\mathcal{F} = \{F_0\}$, put $F_1 = \emptyset$. It follows that $MGR(X \setminus F_1) \subset I$. Since $X \setminus F_1$ is nonempty and open, we get $MGR(X) \subset I$ by invariance of $MGR(X)$ and I under homeomorphisms from H. If there is a set $A \in I \setminus MGR(X)$, then, since I is $\mathbf{\Sigma}^0_2$ supported, we can find $A \in \mathbf{\Sigma}^0_2$, $A \in I \setminus MGR(X)$. Now the Baire Category Theorem implies that there is an open set in I which is impossible. ∎

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